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On the Path Integral of the Relativistic Electron

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We revisit the path integral description of motion of a relativistic electron. Applying a minor, but well-motivated conceptional change to Feynman's chessboard model, we obtain exact solutions of the Dirac equation. The calculation is performed by means of a particularly simple method different from both the combinatorial approach envisaged by Feynman and its Ising model correspondence.

1. INTRODUCTION

It is well known that the continuum propagator of the Dirac equation can be found by summing over random walks. Renewed interest in this issue has arisen in connection with the investigation of stochastic processes which have been shown to be related to the Dirac equation.[3,8] Likewise, the correspondence between the path integral and the Ising model has been explored[4,5] and solutions for a discretized version of the Dirac equation have been found.[6]

As described by Feynman and Hibbs,[1] the propagator of the (1 + 1)-dimensional Dirac equation

$$i\partial\Psi/\partial t = -i\sigma_z\partial\Psi/\partial x - m\sigma_x\Psi \tag{1}$$

(where units $c = \hbar = 1$ are assumed and σ_x and σ_z are the respective Pauli spin matrices) can be found from a model of the one-dimensional motion of a relativistic particle. In this model the motion of the electron is restricted to movements either forward or backward occurring at the speed of light. Assuming units $c = 1$, the motion of the particle corresponds to a sequence of straight path segments of slope $\pm 45°$ in the x–t plane. The retarded propagator $K(x, t)$ of the Dirac equation may then obtained from the limiting process (see, e.g., refs. 1 and 5)

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$$K_{\delta\gamma}(x,\ t) = \lim_{N\to\infty} A_{\delta\gamma}(\epsilon) \sum_{R\geq 0} N_{\delta\gamma}(R)(im\epsilon)^R \tag{2}$$

Here N is the number of segments of constant length $\epsilon = t/N$ of the particle's path between its start point (which is assumed to be the origin of the corresponding coordinate system) and the end point $(x,\ t)$ of the path. R denotes the number of bends, and $N_{\delta\gamma}(R)$ stands for the total number of paths consisting of N segments with R bends. The indices γ and δ correspond to the directions forward or backward at the path's start and end points, respectively, and refer to the components of K. Here $A_{\delta\gamma}(\epsilon)$ accounts for a convenient normalization.

2. MODEL AND CALCULATIONS

In this note we demonstrate that a minor conceptional change of Feynman's chessboard model naturally and directly yields *exact* solutions to the Dirac equation (1). The conceptional change is suggested by the observation that a path with R bends between given start and end points is determined by $R - 1$ bends. For a sketch of the situation consider Fig. 1. The path

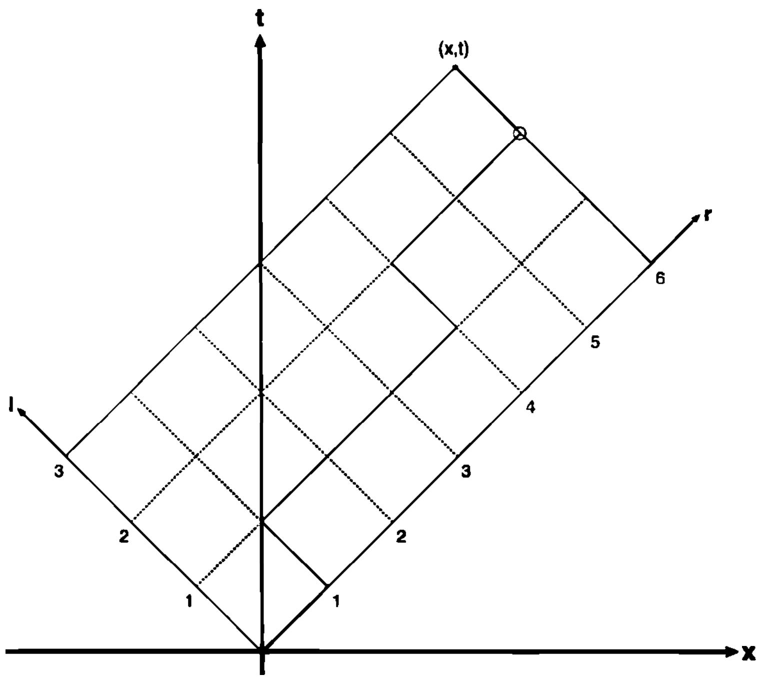

Fig. 1. A possible path with five bends between given start and end points. The first two bends to the right and left, respectively, determine the path since the location and direction of the last bend (indicated by a circle) is fully determined by the first four bends.

shown in Fig. 1 exhibits five bends, three to the left and two to the right. However, the first two bends to the right and left, respectively, determine the path since the location and direction of the last bend (indicated by a circle in Fig. 1) is fully determined by the first four bends. We thus consider here, in contrast to the original formulation of the model, where all bends occurring on a path contribute to the total amplitude, only contributions to the total amplitude from bends which *actually* define the path. In light of the general path integral formalism, it makes perfect sense to consider only those bends which define a path, i.e., the *minimum* information characterizing a path.

In the following we demonstrate by an explicit calculation that the modified model directly leads to exact solutions of the Dirac equation (1). We will use a calculation scheme different from the combinatorial approach envisaged by Feynman and Hibbs[1] and its Ising model correspondence.[4] Following Feynman's chessboard model, we consider each bend which defines a possible path to contribute an amplitude

$$\phi_{j_r} = \mathrm{i}m\epsilon_{j_r} \tag{3}$$

where $\epsilon_{j_r} \doteq \epsilon$ is the length of a path segment. The total amplitude contributed by a path is the product

$$\phi = \prod_r (\mathrm{i}m\epsilon_{j_r}) \tag{4}$$

where j_r runs over all the segments followed by a bend. While the index r enumerates the path segments after which bends occur, the value of j_r indicates the corresponding segment. A path with R bends which starts with positive velocity (i.e., to the right) and ends with negative velocity (i.e., to the left) consists of exactly $(R-1)/2+1$ bends to the left and $(R-1)/2$ to the right. The $(R-1)/2$ bends to the right may occur after any arbitrary path segment to the left. $(R-1)/2$ of the $(R-1)/2+1$ bends to the left occur in the same manner after path segments to the right, while the additional bend to the left occurs after the last segment. Let P be the total number of path segments to the right $(+)$ and Q those to the left $(-)$. Then, the contribution of the $R^+ = (R-1)/2$ bends to the right to Ψ_{-+} is

$$\begin{aligned} \Psi_{-+}(R^+) &= N_{-+}(R^+) \prod_{r=1}^{R^+} (\mathrm{i}m\epsilon_{j_r}) \\ &= \sum_{j_1<\cdots<j_{R^+}}^{P-1} (\mathrm{i}m\epsilon)^{R^+} \end{aligned} \tag{5}$$

For $P \gg 1$, $\Psi_{-+}(R^+)$ is approximated by

$$\Psi_{-+}(R^+) \approx \frac{1}{R^+!} \sum_{j_1 \neq \cdots \neq j_{R^+}}^{P} (\mathrm{i}\epsilon)^{R^+}$$
$$\approx \frac{(\mathrm{i}m\epsilon)^{R^+}}{R^+!} \left(\sum_{j_r=1}^{P} 1 \right)^{R^+}$$
$$= \frac{P^{R^+}(\mathrm{i}m\epsilon)^{R^+}}{R^+!} \tag{6}$$

The contribution of the $R^- = (R-1)/2 + 1 - 1 = (R-1)/2$ bends to the left is calculated similarly. The additional bend (occurring after the last segment to the right) does not enter the calculation since a possible path is fully determined by the location of its $R - 1$ bends to the right and left, respectively. Therefore we find

$$\Psi_{-+}(R^-) \approx \frac{Q^{R^-}(\mathrm{i}\epsilon)^{R^-}}{R^-!} \tag{7}$$

In the limit $N \to \infty$ (i.e., P, $Q \to \infty$) the exact expression for Ψ_{-+} becomes

$$\Psi_{-+} = \sum_{\text{odd } R} (\mathrm{i}m\epsilon)^{R-1} \frac{(PQ)^{(R-1)/2}}{[((R-1)/2)!]^2} \tag{8}$$

where $\epsilon = t/(P+Q)$. With $v = \Delta x/\Delta t = x/t = (P-Q)/(P+Q)$ the classical velocity attributed to the particle, $PQ = [(P+Q)/2\gamma]^2$, where $\gamma = 1/\sqrt{1-v^2}$. Thus we have

$$\Psi_{-+} = \sum_{k=0}^{\infty} (-1)^k \frac{(mt/2\gamma)^{2k}}{[(k)!]^2} = J_0(mt/\gamma) \tag{9}$$

where J_0 is the zeroth-order Bessel function of the first kind. A similar calculation yields for Ψ_{+-} the same result.

For Ψ_{++}, the number of bends to the right and to the left is $R/2$ for each direction where R is even. However, the path is again defined by $R^+ = R/2$ bends to the right and $R^- = R/2 - 1$ bends to the left. Thus,

$$\Psi_{++} = \sum_{\text{even } R} (\mathrm{i}m\epsilon)^{R-1} \frac{P^{R/2}Q^{R/2-1}}{(R/2)!(R/2-1)!}$$
$$= \mathrm{i}\sqrt{P/Q} \sum_{k=0}^{\infty} (-1)^k \frac{(mt/2\gamma)^{2k+1}}{(k+1)!(k)!}$$
$$= \mathrm{i}\sqrt{P/Q} J_1(mt/\gamma) \tag{10}$$

With $\sqrt{P/Q} = (t+x)/(t^2 - x^2)^{1/2}$ and $\tau = (t^2 - x^2)^{1/2}$ the component Ψ_{++} becomes

$$\Psi_{++} = \mathrm{i}(t + x)/\tau\, J_1(mt/\gamma) \tag{11}$$

A similar calculation yields

$$\Psi_{--} = \mathrm{i}(t - \mathrm{x})/\tau\, J_1(mt/\gamma) \tag{12}$$

This completes the envisaged computation. As a side remark note that the presented calculation scheme is not restricted to $\epsilon_{j_r} \doteq \epsilon$. As will be shown elsewhere, similar results may be obtained for $\epsilon_{j_r} = \epsilon(j_r)$.

3. DISCUSSION

To relate the components $\Psi_{\delta\gamma}$ to the solution of the Dirac equation (1), consider the explicit represation

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{13}$$

As may be seen by direct calculation, in this representation Ψ_1 and Ψ_2 defined as

$$\Psi_1 = \begin{pmatrix} \Psi_{++} \\ \Psi_{+-} \end{pmatrix}, \qquad \Psi_2 = \begin{pmatrix} \Psi_{+-} \\ \Psi_{--} \end{pmatrix} \tag{14}$$

are two independent, *exact* solutions of the Dirac equation (1). This completes the demonstration that Feynman's chessboard model yields exact solutions to the Dirac equation when taking into account only those bends which *actually* define paths. With regard to fundamental theories of spacetime and/ or quantum mechanics (e.g., in the spirit of Finkelstein[(2)]) this could be of importance. Similar results have been obtained from the continuum limit of a discretized version of the Dirac equation.[(6)]

The calculation scheme and part of the results presented here can be generalized to unevenly spaced spacetime lattices. This opens up the possibility to define an analog to the Feynman checkerboard for discrete spacetime models of different type (e.g., ref. 7). Related work is in progress and will be presented elsewhere.

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